CHAPTER TWO

Shear force and bending moment of beams

2.1 Beams

A beam is a structural member resting on supports to carry vertical loads. Beams are generally placed horizontally; the amount and extent of external load which a beam can carry depends upon:

a. The distance between supports and the overhanging lengths from supports;
b. The type and intensity of load;
c. The type of supports; and
d. The cross-section and elasticity of the beam.

2.2 Classification of beams

1. Cantilever Beam

A **Built-in** or **encastre'** support is frequently met. The effect is to fix the direction of the beam at the support. In order to do this the support must exert a "fixing" moment M and a reaction R on the beam. A beam which is fixed at one end in this way is called a **Cantilever**. If both ends are fixed in this way the reactions are not statically determinate. In practice, it is not usually possible to obtain perfect fixing and the fixing moment applied will be related to the angular movement of the support. When in doubt about the rigidity, it is safer to assume that the beam is freely supported.

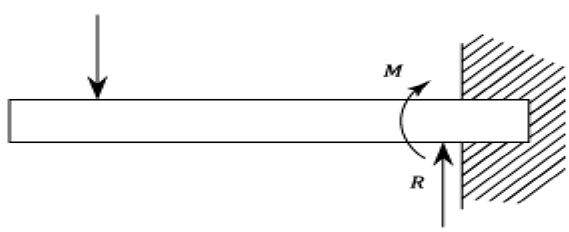


Fig. 8 Cantilever beam

2. Simply Supported Beam

It is a beam having its ends freely resting on supports.

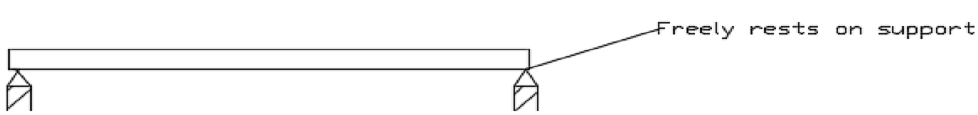


Fig. 9 Simply supported beam

3. Overhanging Beam

A beam having one or both ends extended over supports is known as overhanging beam.

Fig. 10. (i) Overhanging at one end (ii) Overhanging at both ends

4. Propped Cantilever Beam

When a support is provided at some suitable point of a Cantilever beam, in order to resist the deflection of the beam, it is known as propped Cantilever beam.

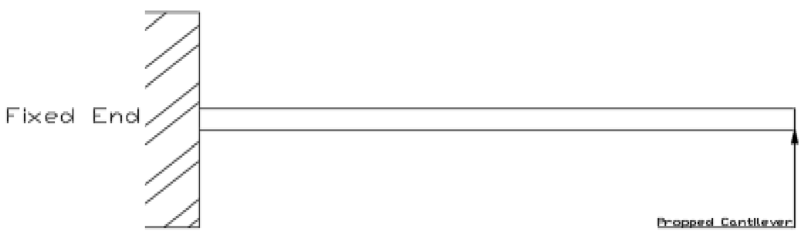


Fig. 11. Propped Cantilever beam

5. Fixed Beam

A beam having its both ends rigidly fixed or built0in to the supporting walls or colums is known as fixed beam.

Fig. 12. Fixed beam

2.3 TYPES OF LOADING

1. Point Load or Concentrated Load

These loads are usually considered to be acting at a point. Practically point load cannot be placed on a beam. When a member is placed on a beam it covers some space or width. But for calculation purpose, we consider the load as transmitting at the central with of the member.

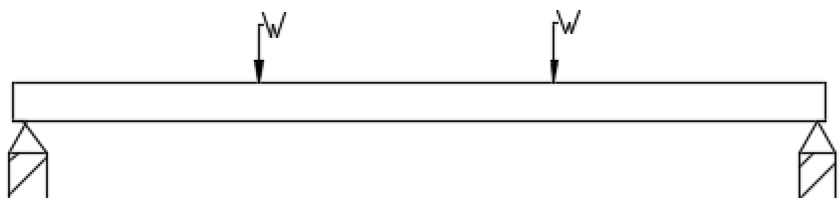


Fig. 13. Concentrated load

2. Uniformly Distributed Load or U.D.L

Uniformly distributed load is one which is spread uniformly over beam so that each unit of length is loaded with same amount of load, and are denoted by Newton/metre.

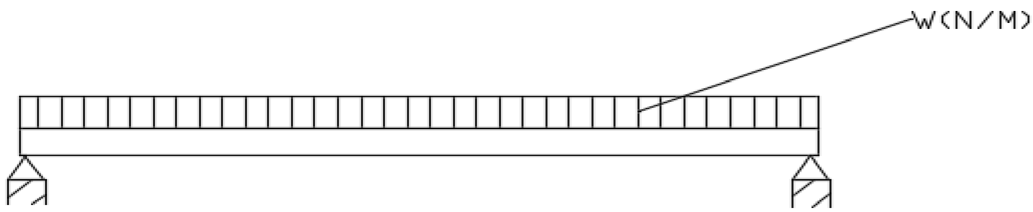


Fig. 14. UDL

3. Gradually Varying Load

If the load is spread, varying uniformly along the length of a beam, then it is called uniformly varying load.

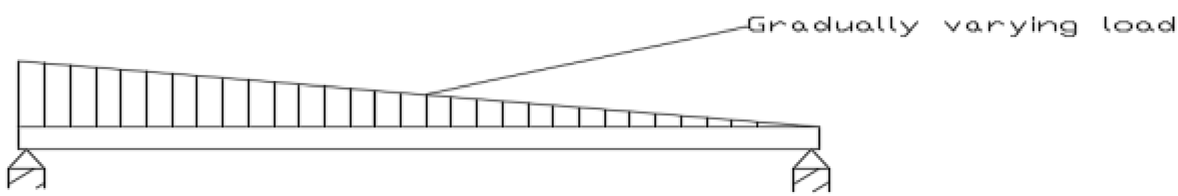


Fig. 15 Gradually varying load

4. Continuous Beam

A beam which rests on more than two supports is known as continuous beam. This may either be overhanging at one or both ends.

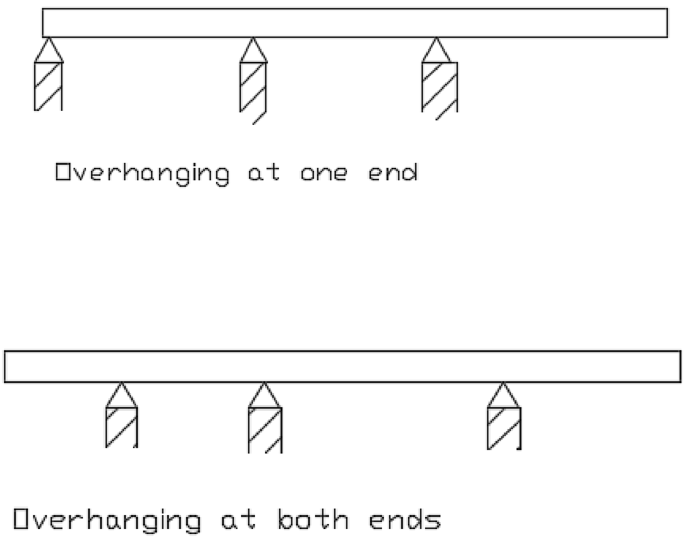


Fig. 16. Overhanging beam

2.4. Span

Clear Span: This is the clear horizontal distance between two supports

Effective Span: This is the horizontal distance between the Centre of end bearings of support.

Effective Span = clear span + oce bearing.

Fig. 17. Effective and clear span

2.5. Shear force

At any section in a beam carrying transverse loads the shearing force is defined as the algebraic sum of the forces taken on either side of the section. Similarly, the bending moment at any section is the algebraic sum of the moments of the forces about the section, again taken **on** either side. In order that the shearing-force and bending-moment values calculated **on** either side of the section shall have the same magnitude and sign, a convenient sign convention has to be adopted. Shearing-force (S.F.) and bending-moment (B.M.) diagrams show the variation of these quantities along the length of a beam for any fixed loading condition.

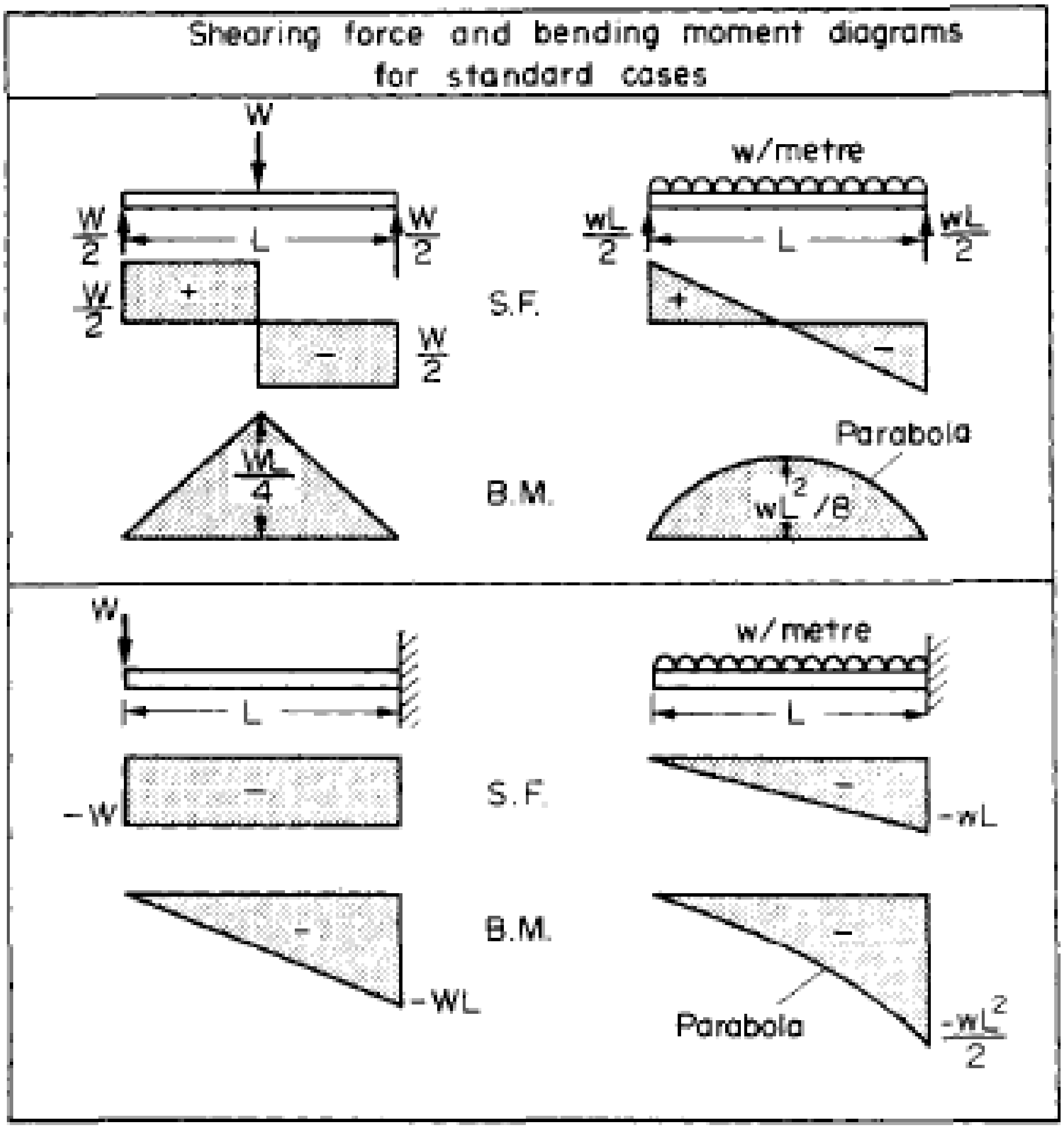


Fig. 18 Shear force and bending moment diagram

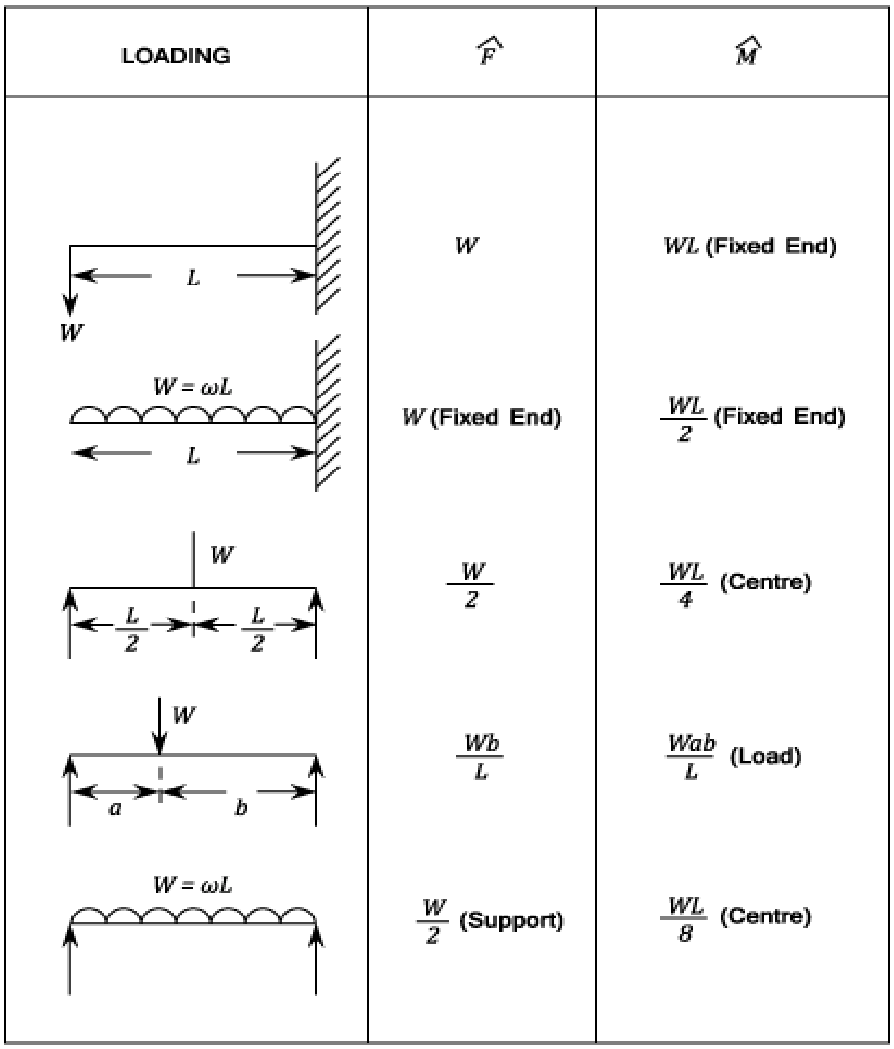


Fig. 19 Loading in beams, shear force and bending moment

At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite, and whose combined action tends to shear the section in one of the two ways. *The shearing force (S.F.) at the section is defined therefore as the algebraic sum of the forces taken on one side of the section.* Which side *is* chosen is purely a matter of convenience but in order that the value obtained on both sides shall have the same magnitude and sign a convenient sign convention has to be adopted.

a. Shearing force (S.F.) sign convention

Forces upwards ***to*** the left of a section or downwards to the right of the section are positive. Thus Fig. --a shows a positive S.F. system at X-X and Fig. ---b shows a negative S.F. system

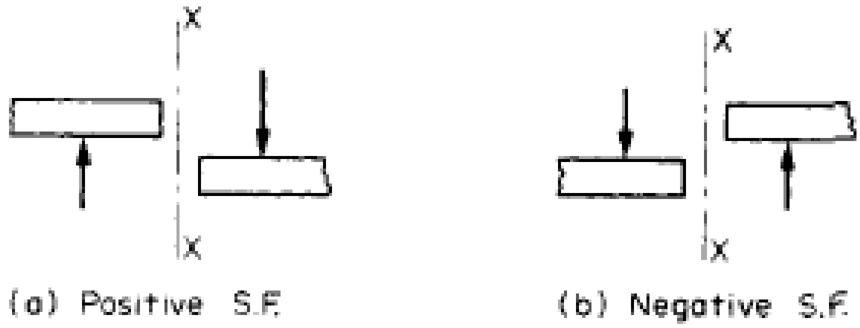


Fig. 20 shear force sign convection

In addition to the shear, every section of the beam will be subjected to bending, i.e. to a resultant **B.M.** which is the net effect of the moments of each of the individual loads. Again, for equilibrium, the values on either side of the section must have equal values. *The bending moment (B.M.) is defined therefore as the algebraic sum of the moments of the forces about the section, taken on either side of the section.* As for *S.F.*, a convenient sign convention must be adopted.

b. Bending moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive. Thus
Fig. a - shows a positive bending moment system resulting in *sagging* of the beam at X-X and
Fig. b- illustrates a negative **B.M.** system with its associated *hogging* beam.

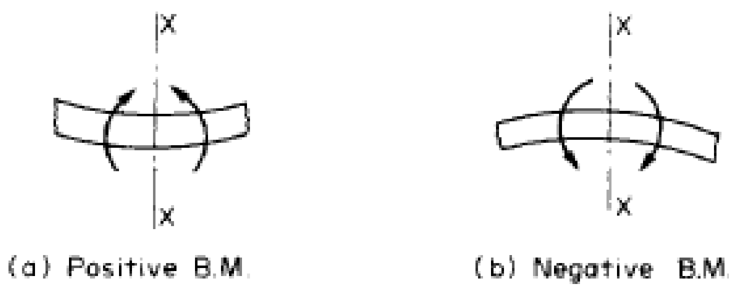


Fig. 21 Bending moment sign cnvection

It should be noted that whilst the above sign conventions for S.F. and **B.M.** are somewhat arbitrary and could be completely reversed, the systems chosen here are the only ones which yield the mathematically correct signs for slopes and deflections of beams in subsequent work and therefore are highly recommended.

c. Shearing force

The shearing force (SF) at any section of a beam represents the tendency for the portion of the beam on one side of the section to slide or shear laterally relative to the other portion. The diagram shows a beam carrying loadsW_1, W_2 and W_3. It is simply supported at two points where the reactions are R_1 and R_2. Assume that the beam is divided into two parts by a section XX. The resultant of the loads and reaction acting on the left of AA is F vertically upwards, and since the whole beam is in equilibrium, the resultant force to the right of AA must be F downwards. F is called the **Shearing Force** at the section AA. It has been defined as *The shearing force at any section of a beam is the algebraic sum of the lateral components of the forces acting on either side of the section.* Where forces are neither in the lateral or axial direction they must be resolved in the usual way and only the lateral components are used to calculate the shear force.

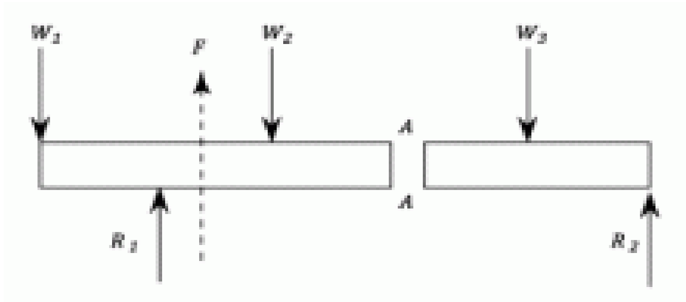


Fig 22 shear force through a section

d. Bending Moment

In a similar manner it can be seen that if the Bending moments (BM) of the forces to the left of AA are clockwise, then the bending moment of the forces to the right of AA must be anticlockwise.

Bending Moment at AA has been defined as the algebraic sum of the moments about the section of all forces acting on either side of the section. Bending moments are considered positive when the moment on the left portion is clockwise and on the right anticlockwise. This is referred to as a **sagging** bending moment as it tends to make the beam concave upwards at AA. A negative bending moment is termed **hogging**.

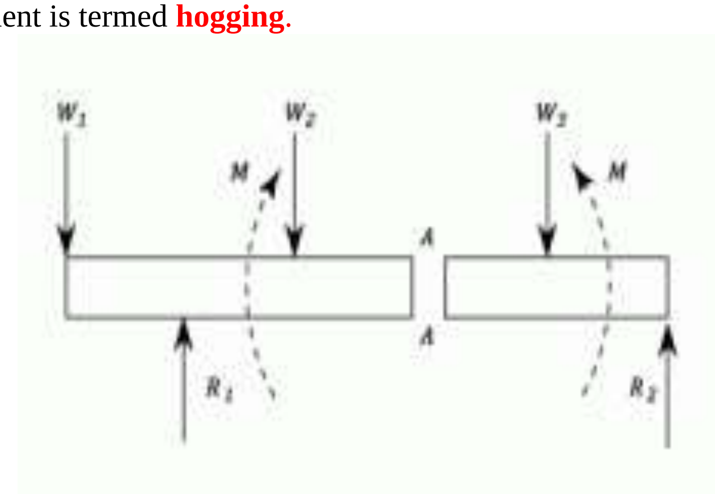


Fig. 23 mendinng moment through a section

RELATIONSHIP BETWEEN LOAD (w), SHEAR FORCE (F), AND BENDING MOMENT (M).

In the following diagram δx is the length of a small slice of a loaded beam at a distance x from the origin O

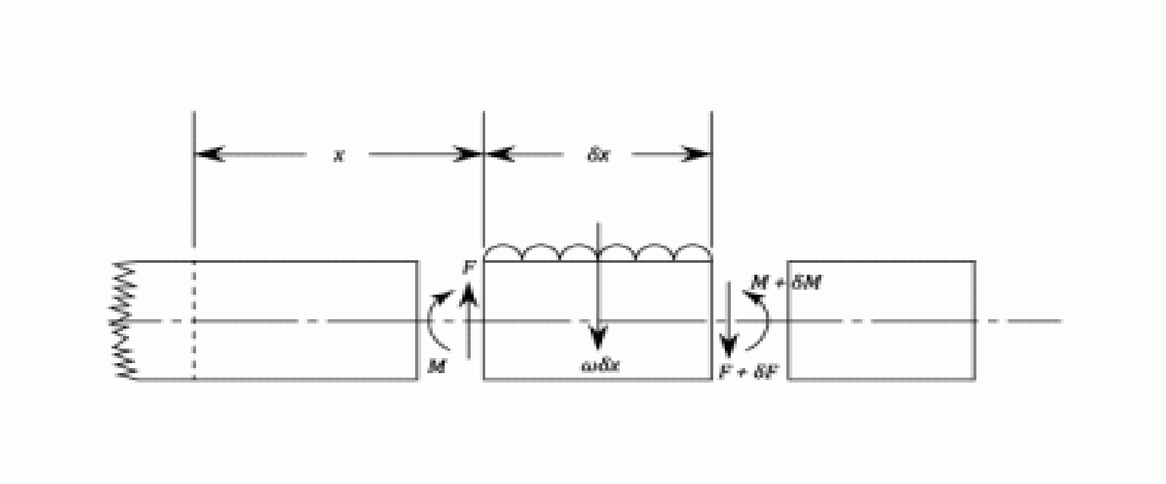


Fig. 24 Loaded beam of length x from origin O

Let the shearing force at the section x be F and at $x = \delta x \;\; be \;\; F + \delta F$. Similarly, the bending moment is M at x, and $M + \delta M \;\; at \;\; x + \delta x$. If w is the mean rate of loading of the length δx then the total load is $w\delta x$, acting approximately (exactly if uniformly distributed) through the centre C. The element must be in equilibrium under the action of these forces and couples and the following equations can be obtained:-

Taking Moments about C:

$$M + F.\frac{\delta x}{2} + (F + \delta F)\,\frac{\delta x}{2} = M + \delta M \qquad \textbf{(1)}$$

Neglecting the product $\delta F.\delta x$ in the limit:

$$F = \frac{dM}{dx} \qquad \textbf{(2)}$$

Resolving vertically:

$$w\delta x + F + \delta F = F \qquad \textbf{(3)}$$

$$Or \qquad w = -\frac{dF}{dx} \qquad \textbf{(4)}$$

$$= -\frac{d^2M}{dx^2} \qquad from\ equation\ (2) \qquad \textbf{(5)}$$

From equation (2) it can be seen that if M is varying continuously, zero shearing force corresponds to either maximum or minimum bending moment. It can be seen from the examples

that "peaks" in the bending moment diagram frequently occur at concentrated loads or reactions, and these are not given by $F = \frac{dM}{dx} = 0$; although they may in fact represent the greatest bending moment on the beam. Consequently, it is not always sufficient to investigate the points of zero shearing force when determining the maximum bending moment.
At a point on the beam where the type of bending is changing from sagging to hogging, the bending moment must be zero, and this is called a point of *inflection* or *contraflexure*.
By integrating equation (2) between the x = a and x = b then:

$$M_b - M_a = \int_a^b F\, dx$$

(6)

Which shows that the increase in bending moment between two sections is the area under the shearing force diagram.
Similarly integrating equation (4)

$$F_a - F_b = \int_a^b w\, dx$$

(7)

equals the area under the load distribution diagram.
Integrating equation (5) gives:

$$M_a - M_b = \int \int_a^b w\, dx.dx$$

(8)

These relations can be very valuable when the rate of loading cannot be expressed in an algebraic form as they provide a means of graphical solution.

2.3 Concentrated Loads

Tutorials.

Problem 1. A Cantilever of length *l* carries a concentrated load *W* at its free end. Draw the Shear Force (SF) and Bending Moment (BM) diagrams.

Solution:

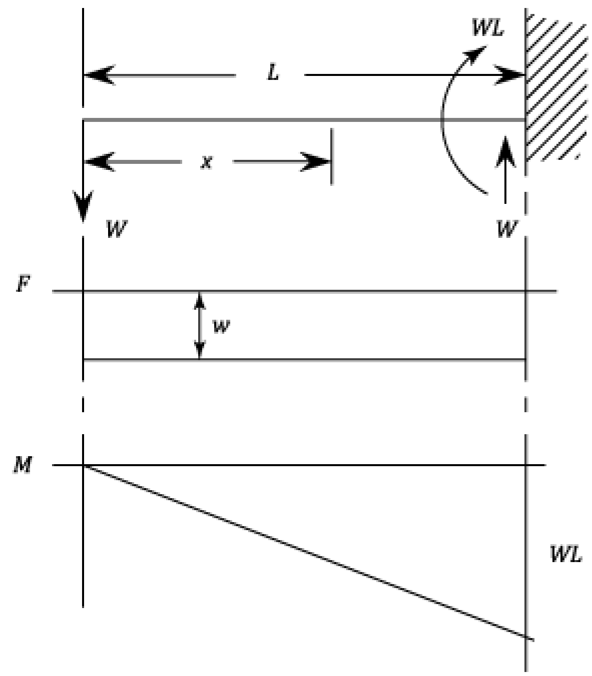
A Cantilever of length *l* carries a concentrated load *W* at its free end. Draw the Shear Force (SF) and Bending Moment (BM) diagrams.

Consider the forces to the left of a section at a distance *x* from the free end.

Then F = - W and is constant along the whole cantilever i.e. for all values of *x*
Taking Moments about the section gives M = - W *x* so that the maximum Bending Moment occurs when *x* = *l* i.e. at the fixed end.

$$\hat{M} = W\, l \quad (Hogging)$$ **(1)**

From equilibrium considerations it can be seen that the fixing moment applied at the built in end is WL and the reaction is W. Hence the **SF** and **BM** diagrams are as follows:



The following general conclusions can be drawn when only concentrated loads and reactions are involved.

- The shearing force suffers sudden changes when passing through a load point. The change is equal to the load.
- The bending moment diagram is a series of straight lines between loads. The slope of lines is equal to the shearing force between the loading points.

2.4 Uniformly Distributed Loads

Problem 2. Draw the SF and BM diagrams for a simply supported beam of length *l* carrying a uniformly distributed load *w* per unit length which occurs across the whole Beam.

Solution.

The Total Load carried is *wl* and by symmetry the reactions at both end supports are each w*l*/2
If *x* is the distance of the section measured from the left-hand support then:

$$F = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right) \tag{2}$$

This give a straight line graph equal to the rate of loading. The end values of Shearing Force are

$$\pm\frac{w\,l}{2}$$

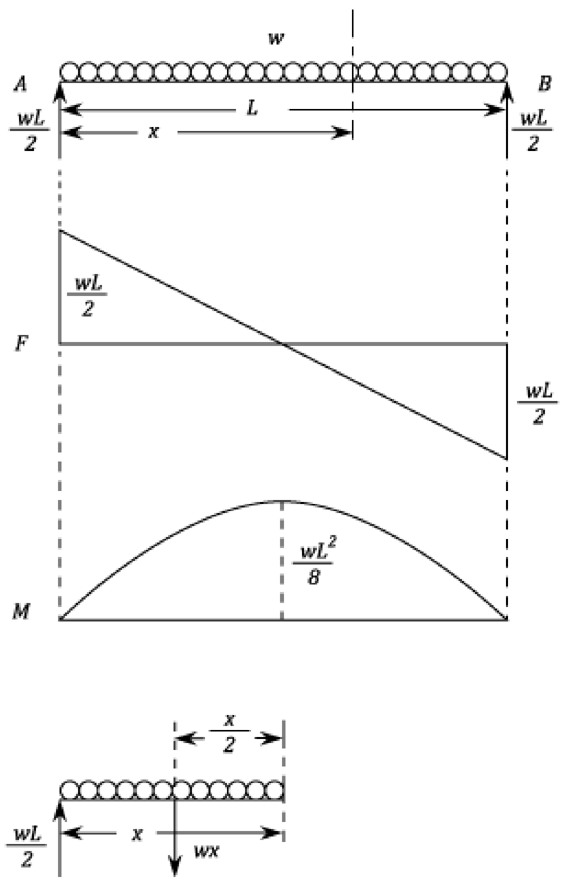
The Bending Moment at the section is found by assuming that the distributed load acts through its center of gravity which is *x/2* from the section.

$$Hence \qquad M = \left(\frac{w\,l}{2}\right)x - (wx)\,\frac{x}{2}$$

(3)

$$= \left(\frac{wl}{2}\right)(l - x)$$

(4)



This is a parabolic curve having a value of zero at each end. The maximum is at the center and corresponds to zero shear force.
From Equation (2)

$$\hat{M} = \left(\frac{w\,l}{4}\right)\left(l - \frac{l}{2}\right)$$

(5)

Putting $x = l/2$

$$\hat{M} = \frac{w\,l^2}{8}$$

(6)

2.5 Combined Load

Problem 3. A Beam 25 m. long is supported at A and B and is loaded as shown. Sketch the SF and BM diagrams and find (a) the position and magnitude of the maximum Bending Moment and (b) the position of the point of contra flexure.

Solution

Taking Moments about B

$$20\,R_a = 10 \times 15 + 2 \times 5 - 3 \times 5$$

(7)

(The distributed load is taken as acting at its centre of gravity.)

$\therefore \quad R_a = 7.25$ k N **(8)**

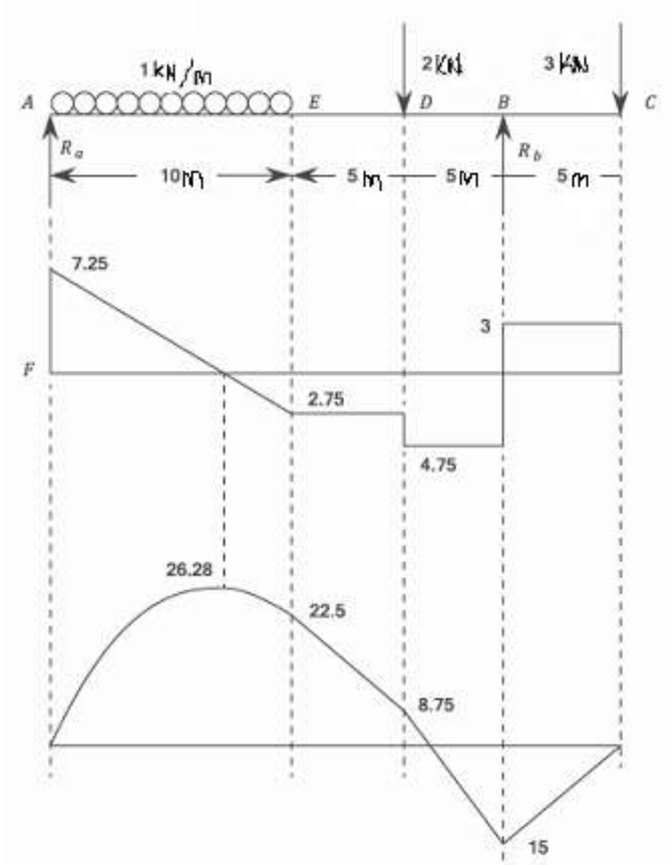
$\therefore \quad R_b =$ Total Load - $R_a = 10 + 2 + 3 - 7.25 = 7.75$ k N **(9)**

The Shearing Force

Starting at A, F = 7.25. As the section moves away from A F decreases at a uniform rate of w per unit length (i.e. f = 7.25 – w x) and reaches a value of - 2.75 at E.

Between E and D, F is constant (There is no load on Ed) and at D it suffers a sudden decrease of 2 kN (the load at D). Similarly there is an increase at B of 7.75 k N (the reaction at B).

This results in a value of F = 3k N at B which remains constant between B and C. Note this value agrees with the load at C.



Bending Moment from A to E:

$$M = R_a\, x - \frac{w\, x^2}{2} = 7.25\, x - \frac{x^2}{2} \quad since \quad (x = 1) \tag{10}$$

This is a parabola which can be sketched by taking several values of x. Beyond E the value of x for the distributed load remains constant at 5 ft. from A
Between E and D

$$M = 7.25\, x - 10(X - 5) \ = -2.75x + 50 \tag{11}$$

This produces a straight line between E and D. Similar equations apply for sections DB and BC. However it is only necessary to evaluate M at the points D and B since M is zero at C. The diagram consists in straight lines between these values.
At D

$$\mathrm{M} = -2.75 \ \times 15 + 50 = 8.75 \ \mathrm{kN.m} \tag{12}$$

At B

$$\mathrm{M} = -3 \ \times 5 = -15 \ \mathrm{kN.m} \tag{13}$$

This last value was calculated for the portion BC
We were required to find the position and magnitude of the maximum BM. This occurs where the shearing force is zero.
i.e. .at 7.25 m. from A

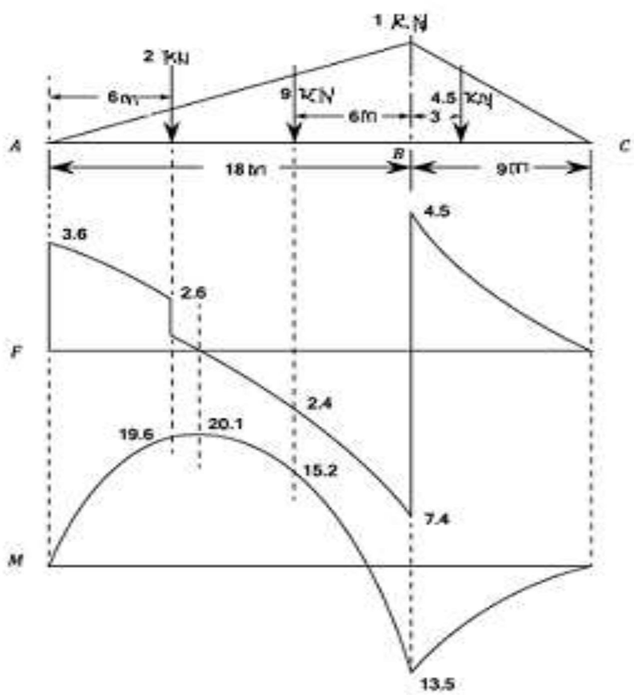
$$\therefore \quad \mathrm{M} = 7.25 \times 7.25 - \frac{7.25^2}{2} = 26.28\ \mathrm{k\,N.m} \tag{14}$$

The point of contraflexure occurs when the bending moment is zero and this is between D and B at:

$$\left(\frac{15}{15+8.75}\right) \times 5 = 3.16\ \mathrm{m\ from\ B} \tag{15}$$

2.6 Varying Distributed Loads

Problem
A Beam ABC, 27m long, is simply supported at A and B 18 m. across and carries a load of 2 kN at 6 m. from A together with a distributed load whose intensity varies in linear fashion from zero at A and C to 1 kN/m. at B.
Workings
Draw the Shear Force and Bending Moment diagrams and calculate the position and magnitude of the maximum B.M.
The Total Load on the beam (i.e. the load plus the mean rate of loading of 1/2 kN/m) is given by: $Load = 2 + \frac{1}{2} \times 27 = 15.5$ kN



The Total distribute load on $AB = \frac{1}{2} \times 18 = 9\ k\ N$ and on $BC = \frac{1}{2} \times 9 = 4.5\ k\ N$ each of which act through their centres of gravity. These are $\frac{2}{3} \times 18 = 12\ m$ from A and $\frac{2}{3} \times 9 = 6\ m$ from C in the other case.

(**Note**. These are the centroids of the triangles which represent the load distribution)

Taking Moments about B

$$R_1 = \left(\frac{2\times12+9\times6-4.5\times3}{18} \right) = 3.6\ \text{kN} \qquad (\mathbf{16})$$

$$\therefore \quad R_2 = 2 + 9 + 4.5 - 3.6 = 11.9\ \text{k N} \qquad \mathbf{(17)}$$

At a distance *x* (<18) from A the loading is x/18 tons/ft.. The Total distributed load on this length is:

$$(\text{Mean rate of loading}) \times = \frac{1}{2}\left(\frac{x}{18}\right)x = \frac{x^2}{36} \ \text{kN} \qquad \mathbf{18)}$$

The centre of gravity of this load is $\frac{2}{3}x$ from A. **For 0<x<6**

$$F = 3.6 - \frac{x^2}{36}$$

(19)

At x = 6 m

F =2.6 kN

(20)

$$M = 3.6\,x - \left(\frac{x^2}{36}\right) \times \frac{x}{3} = 3.6\,x - \frac{x^3}{108}$$

(21)

At x = 6 m

M = 19.6 KN m.

(22)

6< x <18

$$f = 3.6 - 2 - \frac{x^2}{36}$$

$$At\ x = 12\ kN. \qquad F = -2.4\ kN$$

(23)

$$At\ x = 18kN. \qquad F = -7.4\ kN$$

$$F = 0 \; when \; x = 6\sqrt{1.6} = 7.58 \text{ K N.} \quad \textbf{(24)}$$

$$M = 3.6x - 2(x-6) - \frac{x^3}{108} = 1.6x + 12 - \frac{x^3}{108} \quad \textbf{(25)}$$

$$At \; x = 12 \; k \; N. \quad M = 15.2 \text{ k N m.} \quad \textbf{(26)}$$

$$At \; x = 12 \; k \; N. \quad M = -13.5 \text{ k N m}$$

The maximum Bending Moment occurs at zero s earing force 1.e.x = 7.58 kN.

$$M = 20.1 \; kN \; m.$$

The section BC can be more easily calculated by using a variable X measured from C. Then by a similar argument:-

$$F = \frac{1}{2}\left(\frac{x}{9}\right)x = \frac{x^2}{18} \text{ k N.} \quad \textbf{(27)}$$

At x = 9 kN. F = 4.5 k N.

$$M = -\frac{1}{2}\left(\frac{x}{9}\right).x.\left(\frac{x}{3}\right) = -\left(\frac{x^3}{54}\right) \; kN.m \quad \textbf{(28)}$$

$$At \; x = 9 \; kN \quad M = -13.5 \; kN.m \quad \textbf{(29)}$$

The complete diagrams are shown. It can e seen that for a uniformly varying distributed load, the Shearing Force diagram consists of a series of parabolic curves and the Bending Moment diagram is made up of "cubic" discontinuities occurring at concentrated loads or reactions. It has been shown that Shearing Forces can be obtained by integrating the loading function and Bending Moment by integrating the Shearing Force, from which it follows that the curves produced will be of a successively "higher order" in x (See equations (6) and(7))

2.7 Graphical Solutions

This method may appear complicated but whilst the proof and explanation is fairly detailed, the application is simple and straight forward. The change of Bending Moment can be given by the double Integral of the rate of loading. This integration can be carried out by means of a **funicular polygon**. Suppose that the loads carried on a simply supported beam are W_1, W_2, W_3, and W_4; and that R_1 and R_2 are the reactions at the supports. Letter the spaces between the loads and reactions A, B, C, D, E, and F.

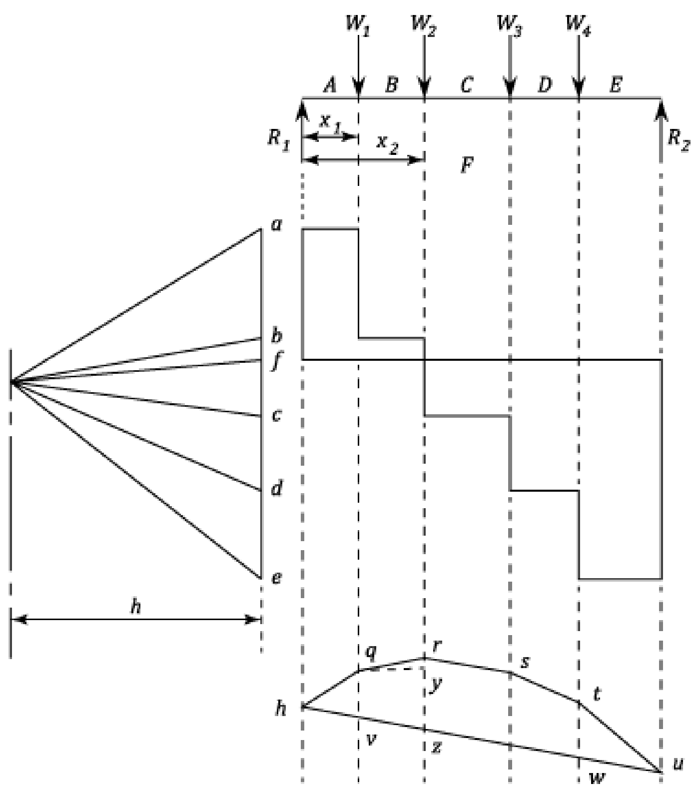
i. Draw to scale a vertical line such that:

$$a\,b = W_1; \quad b\,c = W_2; \quad c\,d = W_3; \quad and \quad d\,e = W_4. \quad \textbf{(1)}$$

ii. Now take any point "O" to the left of the line and join O to a, b, c, d, and e. This is called **The Polar Diagram**

iii. Commencing at any point p on the line of action of R_1 draw **pq** parallel to **Oa** in the space "A" , qr parallel to **Ob** in the space "B" and similarly rs, st, and tu. Draw Of parallel to pu. $W_1, W_2, W_3, and W_4, and that R_1 \; and \; R_2$

It will now be shown that **fa** represents R_1. Also, **pqrstu** is the Bending Moment diagram drawn on a base pu, M being proportional to the vertical ordinates. W_1 is represented by ab and acts through the point q; it can be replaced by forces **a O** along qp and Ob along qr.
Similarly, W_2 can be replaced by forces represented by **b O** along rq and Oc along rs, W_3 by **c O** along **s r** and **O d** along **s t** etc. All of these forces cancel each other out except **aO** along **qp** and **Od** along **te**, and these two forces must be in equilibrium with $R_1 \; and \; R_2$. This can only be so if R_1 is equivalent to a force **Oa** along **pq** and **fO** along up, R_2 being equivalent to **eO** along ut and **Of** along **pu**. Hence, R_1 is represented by **fa** and R_2 by **ef.**



triangles **pqv** and **Oaf** are similar and hence:

$$q\,v = af.\frac{pv}{Of} \tag{2}$$

$$Or \qquad qv \propto a\,f\;\frac{x_1}{h}$$

Where x_1 is the distance from W_1 from the left-hand end of the beam, and h is the length of the perpendicular from **O** on to **ae**. But $ay\,.\,x_1 \;\propto\; R_1\,x_1 \quad i.e.\, the\; BM\; at\; x_1$

Hence for a given position of the pole **O**, **qv** represents the B>M> at x_1 to a certain scale.

If **qy** is drawn parallel **tp pu**, then the triangle **qry** is similar to **Obf** and:

$$r\,y = b\,f\left(\frac{qy}{of}\right) = b\,f\left(\frac{x_2 - x_1}{h}\right) \tag{3}$$

$$\therefore \qquad rz = qv + ry = af\;\left(\frac{x_1}{h}\right) + bf\left(\frac{x_2 \; - x_1}{h}\right) \tag{4}$$

$$which\ is\ \propto R_1x_1 = (R_1 - W_1)(x_2 - x_1) = R_1x_2 - W_1(x_2 \; - X_1) \tag{5}$$

Which is the Bending Moment at x_2.

Similarly, the ordinates at the other load points give the Bending Moments at those points, the scale being determined as follows: If the load scale of the Polar Diagram is 1cm = S_1 kN then the length scale along the beam is,. and the Bending Moment scale required is 1cm = S_3 kN m, then the length

$$qv \propto af.\frac{x_1}{h} \text{ as shown above } = \frac{W_1x_1}{s_1\,s_2h} = \frac{M}{s_1\,s_2\,h} \tag{6}$$

But,

$$qv = \frac{M}{s_3}$$

$$\therefore \qquad h = \frac{s_3}{s_2\,s_2}\;in. \tag{7}$$

If a base on the same level as f is drawn and the points a, b, c, d, and e are projected across from the Polar Diagram, then the Shearing Force diagram is obtained.
This method can be equally well used for distributed loads by dividing the loading diagram into strips and taking the load on a strip to act as if it were concentrated at its centre of gravity.
For cantilevers, if the Pole O is taken on the same horizontal level as the point a, then the base of the Bending Moment will be horizontal.

Shearing Force F

$$F = \frac{dM}{dx} \tag{8}$$

Mending Moment M

Rate of loading w

$$w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2} \tag{9}$$